Gaining or Losing Perspective for Convex Multivariate Functions

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Joint work with Jon Lee (University of Michigan)

Background

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Off
- $x = 0$
- No cost

On
- Operating range $x \in [\ell, u]$
- Fixed cost + Operating cost $f(x)$

Indicator vairable z: $z \in \{0, 1\}$.

Sparse portfolio selection

Mean-variance optimization

- Given n assets with expected return a and covariance matrix Q
- Each asset i has minimum and maximum buy-in thresholds $[\ell_i, u_i]$, and the number of assets to buy is limited by k
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Gaining perspective

Mean-variance optimization

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& v \geq \|w\|^2, w = L^T x, \\
& y_i \geq x_i^2.
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(Frangioni and Gentile [2006], Günlük and Linderoth [2010], D'Ambrosio, Frangioni, and Gentile [2019], Kronqvist, Misener, and Tsay [2022], Wei, Gómez, and Küçükyavuz [2022], Han and Gómez [2024], Shafiee and Kılınç-Karzan [2024] ...)

Gaining perspective

- Perspectivizing $y \geq f(x)$ to $y \geq \tilde{f}(x, z)$, where

$$\tilde{f}(x, z) = \begin{cases} zf(x/z), & z > 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

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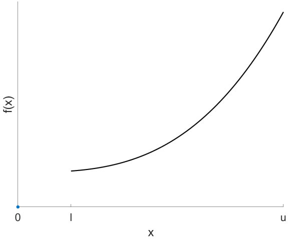

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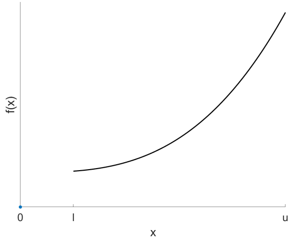

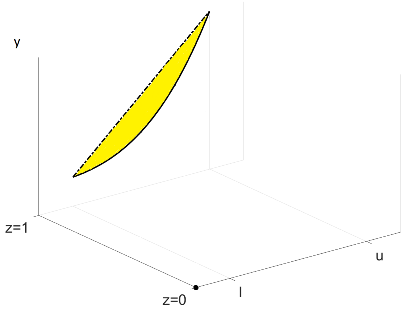

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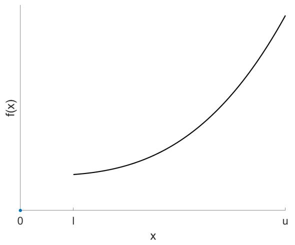

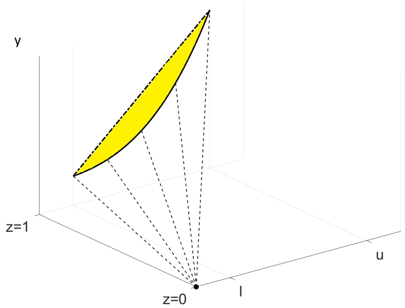

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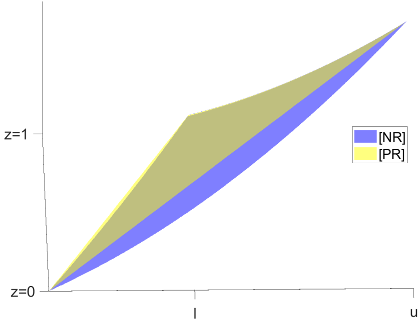
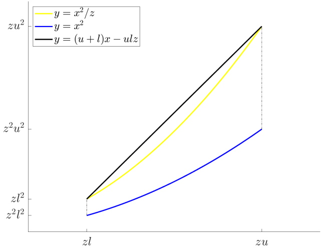
- Conic solvers (like MOSEK and SDPT3) can handle well-known classes of cones: second-order cones, power cones, exponential cones. But they may not handle the other constraints in the model.
- So a main interest of ours is in determining when natural and simple non conic-programming relaxations may be adequate.

Compare relaxations





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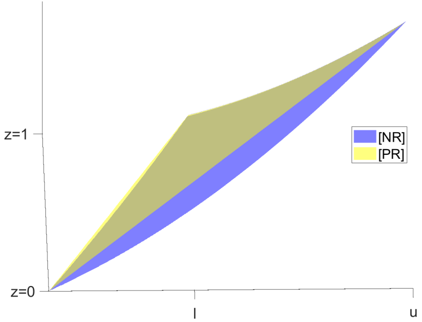

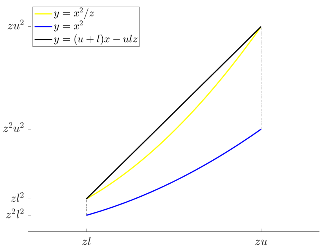

Volume!

Prior work

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- The univariate case will help analyze the separate functions $\sum_{i=1}^{n} f(x_i)$.
- There are many common nonseparable functions: $(c^\top x)^n$, $e^{c^T x}$, $\log\left(\frac{1}{d}\sum_{j=1}^{d} e^{x_j}\right)$.

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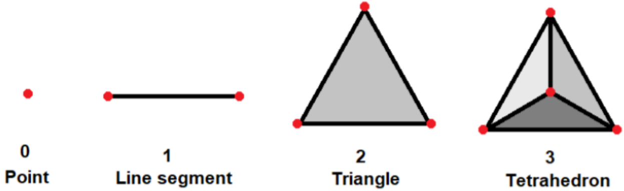
- To handle the multivariate case ($x \in \{0\} \cup Q$), the starting point is to consider $f(x)$ on a simplex J.



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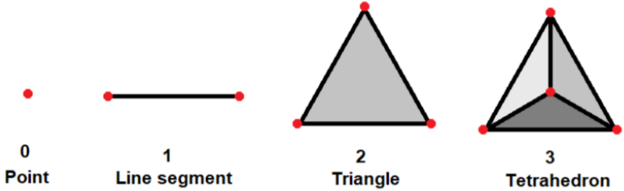

- We will also discuss $f(x)$ on box domains to explain the challenges.

For $x \in \{\mathbf{0}\} \cup J$, we have $f(x)$ with $f(0) = 0$, but convex on $J \subset \mathbb{R}^d_{\geq 0} \setminus \{\mathbf{0}\}$, where $J := \mathrm{conv}\{v_0, v_1, \ldots, v_d\}$.
We define the **disjunctive set**

$$D(f, J) := \{\mathbf{0}_{d+2}\} \cup \{(x, y, 1) \in \mathbb{R}^{d+2} \ : \ \mu(x) \geq y \geq f(x), \ x \in J\}.$$

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$P(f, J) :=$

$$\left\{(x, y, z) \in \mathbb{R}^{d+2} \ : \ z\mu(x/z) \geq y \geq zf(x/z), \ x \in z \cdot J, \ 1 \geq z \geq 0\right\}.$$

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Assuming f is convex on $\mathrm{conv}(\{\mathbf{0}\} \cup J) = \{z \cdot J : 0 \leq z \leq 1\}$.

Naïve relaxation

$P^0(f, J) :=$

$$\left\{(x, y, z) \in \mathbb{R}^{d+2} \;:\; z\mu(x/z) \geq y \geq f(x),\; x \in z \cdot J,\; 1 \geq z \geq 0\right\}.$$

Upper bound

- Upper bound $\mu(x)$ satisfying $\mu(v_j) = f(v_j)$,

$$\mu(x) := w^\top B^{-1}(x - v_0) + f(v_0),$$

where

$$w^\top := [f(v_1) - f(v_0),\ \ldots\ , f(v_d) - f(v_0)] \in \mathbb{R}^{1\times d},$$
$$B := [v_1 - v_0,\ \ldots\ , v_d - v_0] \in \mathbb{R}^{d\times d}.$$

Therefore, the perspective

$$z\mu(x/z) = w^\top B^{-1}(x - zv_0) + f(v_0)z\ .$$

The well-known volume formula for the d-simplex $J := \operatorname{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d$ is

$$\operatorname{vol}(J) = \int_J 1 dx = \frac{1}{d!} \left| \det \begin{bmatrix} v_0 & v_1 & \ldots & v_d \\ 1 & 1 & \ldots & 1 \end{bmatrix} \right| .$$

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Theorem (Perspective relaxation)

$$\mathrm{vol}(P(f,J)) = \frac{\mathrm{vol}(J)}{(d+2)(d+1)} \sum_{j=0}^{d} f(v_j) - \frac{1}{d+2}\int_J f(x)dx.$$

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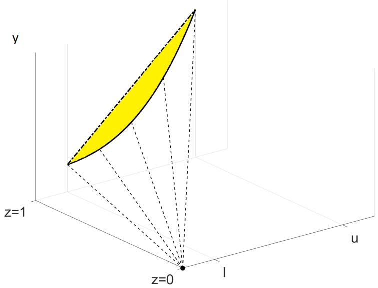

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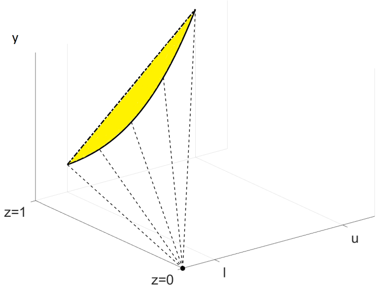

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Corollary

$$\Delta(P^0, P) = \frac{1}{d+2} \int_J f(x)dx - \int_0^1 z^d \int_J f(zx)dxdz.$$

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- $\Delta(P^0, P) \geq 0$ is independent of the upper bound μ.
- These volume formulae are true for the general polytope Q.

q-homogeneous function

$f(x)$ is q-homogeneous if $f(\lambda x) = \lambda^q f(x)$ for $\lambda \geq 0$.

Theorem (Convex q-homogeneous function on $\mathrm{conv}(J \cup \{\mathbf{0}\})$)

$$\int_0^1 z^d \int_J f(zx)dxdz = \frac{1}{q+d+1} \int_J f(x)dx.$$

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Corollary (Lee, Skipper, and Speakman [2021])

For $d = 1$, $J = [\ell, u]$, and $f(x) = x^q$,

$$\Delta(P^0, P) = \frac{(q-1)(u^{q+1} - \ell^{q+1})}{3(q+2)(q+1)}.$$

Power of linear form

Example: $f(x) = x^\top C x$ $(C \succeq 0)$, $f(x) = \sum_{s=1}^{\ell} \lambda_s (c_s^\top x)^q$ $(\lambda_s > 0)$

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- $q > 1$, $f(x)$ is convex with the further assumption $c^\top v_j \geq 0$.
- $q = 2r$ is even, $f(x)$ is convex.

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Consider the cut-off ratio

$$\frac{\Delta(P^0, P)}{\mathrm{vol}(P^0(f, J))}$$

over $J = \mathrm{conv}\{v_0, v_1, \ldots, v_d\}$. Note that the assumption $c^\top v_j \geq 0$ is equivalent to $c \geq 0$ when $J = \Delta_d := \mathrm{conv}\{\mathbf{0}, e_1, \ldots, e_d\}$.

$q > 1$, $c^\top v_j \geq 0$

Theorem

$$\frac{\Delta(P^0, P)}{\mathrm{vol}(P^0(f, J))} \geq \frac{q-1}{\frac{\Gamma(q+d+2)}{(d+1)!\Gamma(q+1)} - (d+2)} \sim O\left(\frac{1}{q^d}\right), \text{ for fix } d$$

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Lemma

$$\int_J f(x)dx \geq d! \,\mathrm{vol}(J) \frac{\Gamma(q+1)}{\Gamma(q+d+1)} \sum_{j=0}^{d} f(v_j).$$

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Lemma

$$\int_J f(x)dx \geq d!\,\mathrm{vol}(J)\frac{\Gamma(q+1)}{\Gamma(q+d+1)}\sum_{j=0}^{d} f(v_j).$$

- The lower bound becomes tight when $\frac{c^\top v_j}{c^\top v_k} \to 0$ for all $j \neq k$, where $c^\top v_k = \max_j c^\top v_j$.
- When $d = 1$, the lower bound recovers $\frac{2}{q+4}$.

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Goal: find a lower bound on the ratio $\int_{\Delta_d} (c^\top t)^q dt$ divided by $\sum_{j=1}^d c_j^q$.

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$$\int_{\Delta_d} (c^\top t)^{2r} dt = \frac{(2r)!}{(2r+d)!} \sum_{\|\mathbf{k}\|_1 = 2r} c_1^{k_1} \dots c_d^{k_d} =: \frac{(2r)!}{(2r+d)!} h_{2r}(c).$$

- $h_{2r}(c)$ is the sum of all monomials with coefficient 1 and degree $2r$, which is called **complete homogeneous symmetric polynomial**.

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Goal: find a lower bound on the ratio $\int_{\Delta_d} (c^\top t)^q dt$ divided by $\sum_{j=1}^d c_j^q$.

$$\int_{\Delta_d} (c^\top t)^{2r} dt = \frac{(2r)!}{(2r+d)!} \sum_{\|\mathbf{k}\|_1 = 2r} c_1^{k_1} \dots c_d^{k_d} =: \frac{(2r)!}{(2r+d)!} h_{2r}(c).$$

- $h_{2r}(c)$ is the sum of all monomials with coefficient 1 and degree $2r$, which is called **complete homogeneous symmetric polynomial**.

- Find a lower bound for

$$\min\{h_{2r}(t) : \sum_{j=1}^{d} t_j^{2r} = 1\}.$$

$q = 2r$ even

Theorem

$$\frac{\Delta(P^0, P)}{\mathrm{vol}(P^0(f, J))} \geq \frac{q-1}{\frac{(q+d+1)!}{(d+1)!q!}2^r r! - (d+2)}$$

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Proof.

- Hunter [1977] gives the bound $h_{2r}(c) \geq \frac{1}{2^r r!} (\sum_{j=1}^{d} c_j^2)^r$.
- $(\sum_{j=1}^{d} c_j^2)^r \geq \sum_{j=1}^{d} c_j^{2r}$.
- $$\int_J f(x)dx \geq d!\,\mathrm{vol}(J) \frac{q!}{(q+d)!} \frac{1}{2^r r!} \sum_{j=0}^{d} f(v_j).$$

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- It is still an open question whether this bound is tight or not.

Improvement on the ratio

Theorem

For (i) $d = 2$, (ii) $q = 2$, (iii) $q = 4$ and $d = 3$, we have

$$h_q(c_1, \ldots, c_d) \geq \frac{1}{2} \sum_{j=1}^{d} c_j^q.$$

The equality holds if and only if $\sum_{j=1}^{d} c_j = 0$.

- $h_2(c_1, \ldots, c_d) - \frac{1}{2} \sum_{j=1}^{d} c_j^d = \frac{1}{2} (\sum_{j=1}^{d} c_j)^2$.
- $h_4(c_1, c_2, c_3) - \frac{1}{2}(c_1^4 + c_2^4 + c_3^4) = \frac{1}{2}(c_1 + c_2 + c_3)^2(c_1^2 + c_2^2 + c_3^2)$.
- $h_6(1, 1, -2)/(1^6 + 1^6 + (-2)^6) = 31/66 \approx 0.4697$.
- $h_4(0.3577, 0.3577, 0.3577, -0.9875)/(c_1^4 + c_2^4 + c_3^4 + c_4^4) \approx 0.4598$.

A class of exponential functions

Example: $f(x) = \exp(\mathbf{1}^\top x) - 1$ (not q-homogeneous)

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Consider the asymptotic behavior of the cut-off ratio

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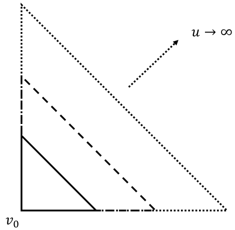

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- Asymptotically, with respect to u, the fraction of the volume of the naïve relaxation that is "extra" beyond the perspective relaxation tends to 0 rather quickly as u tends to ∞.

Box domain case

$x \in \{\mathbf{0}\} \cup Q$, where $Q = A[0,1]^n + b$, and $\operatorname{rank}(A) = n \leq d$.

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Recall Theorem

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$$\Delta(P^0, P) = \frac{1}{d+2}\int_Q f(x)dx - \int_0^1 z^d \int_J f(zx)dxdz.$$

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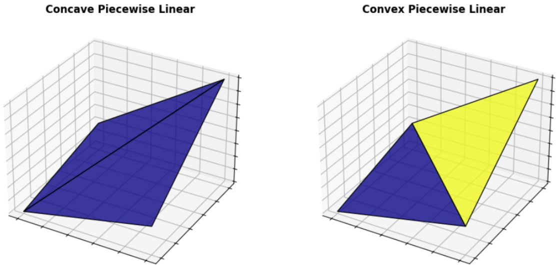
Challenge: The best concave upper bound $\mu(x)$ is no longer a hyperplane.

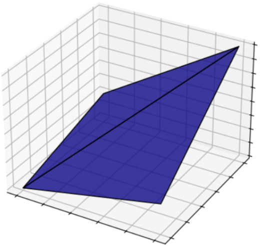


Concave envelope

A function $f(x) : S \mapsto \mathbb{R}$ is said to be *supermodular* if $f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$ for all $x, y \in S$, where $x \vee y$ and $x \wedge y$ denotes the elementwise maximum and minimum of x and y.

Theorem (Tawarmalani, Richard, and Xiong [2013])

If $f : [0,1]^d \mapsto \mathbb{R}$ is ***supermodular*** *when restricted to $\{0,1\}^d$, then the concave envelope of f over $[0,1]^d$ is given by the piecewise linear function through Kuhn's triangulation $\Delta_{i_1,\dots,i_d} := \{x : 0 \leq x_{i_1} \leq \cdots \leq x_{i_d} \leq 1\}$, for any permutation $(i_1, \dots, i_d)$ of $[d]$.*



$\operatorname{conc}(f)$

Box domain case

- $f(x) = e^{c^\top x} - 1 (c \geq 0)$ is supermodular.
- We consider two natural upper bound functions μ and compare the asymptotic behavior of the cut-off ratio on a scaled box $Q_u := v_0 + u[0,1]^d$
  1. μ is a best constant upper bound $F := \max_{x \in Q}\{f(x)\}$;
  2. μ is $\mathrm{conc}(f)$.

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  1. μ is a best constant upper bound $F := \max_{x \in Q}\{f(x)\}$;
  2. μ is $\operatorname{conc}(f)$.

Theorem

$$\lim_{u \to \infty} \frac{\operatorname{vol}(P^0(f, Q_u, F))}{\operatorname{vol}(P^0(f, Q_u, \operatorname{conc}(f)))} = d + 1.$$

$$\lim_{u \to \infty} u^d \cdot \frac{\Delta(P^0, P)}{\operatorname{vol}(P^0(f, Q_u, \operatorname{conc}(f)))} = \frac{d+1}{\prod_{j=1}^{d} c_j}.$$

Box domain case

- $f(x) = (c^\top x)^q (q > 1, c \geq 0)$ is supermodular.

$$\frac{\Delta(P^0, P)}{\mathrm{vol}(P^0(f, Q_u, \mathrm{conc}(f)))} \geq \frac{\Delta(P^0, P)}{\mathrm{vol}(P^0(f, Q_u, F))}$$

Theorem

$$\lim_{u \to \infty} \frac{\Delta(P^0, P)}{\mathrm{vol}(P^0(f, Q_u, F))} > 0.$$

Conclusion & Future Work

- We give a way to calculate the volume of the perspective and naïve relaxation
- We analyze the cut-off ratio for several important classes of functions on a simplex and box domains with different behaviors

Conclusion & Future Work

- We give a way to calculate the volume of the perspective and naïve relaxation
- We analyze the cut-off ratio for several important classes of functions on a simplex and box domains with different behaviors
- To understand the asymptotic behavior of the cut-off ratio in terms of more general classes of functions and domains.
- To explore the affects of triangulation over a polytope on these relaxations
- To incorporate into efficient algorithms to solve problems in application

Thank You!

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Integration over a simplex

Monomial formula over a standard simplex

$$\int_{\Delta_d} x_1^{\alpha_1} \dots x_d^{\alpha_d} dx = \frac{\prod_{j=1}^{d} \alpha_j!}{(d + \sum_{j=1}^{d} \alpha_j)!},$$

where $\alpha_j \in \mathbb{Z}_{\geq 0}$, $j = 1, \dots, d$.

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where $\alpha_j \in \mathbb{Z}_{\geq 0}$, $j = 1, \dots, d$.

More generally,

$$\int_{\Delta_d} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d} (1 - x_1 - \dots - x_d)^{\alpha_{d+1}} dx = \frac{\prod_{j=1}^{d+1} \Gamma(\alpha_j + 1)}{\Gamma(\sum_{j=1}^{d+1} \alpha_j + d + 1)},$$

where $\alpha_j \in \mathbb{R}$, $\alpha_j > -1$, and the gamma function $\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} dx$ for $z > 0$.

Back

Integration over a simplex

Theorem (Baldoni, Berline, De Loera, Köppe, and Vergne [2010])

$$\int_J (c^\top x)^n dx = d!\,\mathrm{vol}(J)\frac{n!}{(n+d)!}\sum_{\mathbf{k}\in\mathbb{Z}^{d+1}_{\geq 0}, \|\mathbf{k}\|_1 = n} (c^\top v_0)^{k_0}\dots(c^\top v_d)^{k_d}\ ,$$

$$\int_J e^{c^\top x} dx = d!\,\mathrm{vol}(J)\sum_{\mathbf{k}\in\mathbb{Z}^{d+1}_{\geq 0}} \frac{(c^\top v_0)^{k_0}\dots(c^\top v_d)^{k_d}}{(\|\mathbf{k}\|_1 + d)!}.$$

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In particular, when $J = \Delta_d$, we have

$$\int_{\Delta_d} (c^\top x)^n dx = \frac{n!}{(n+d)!} \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^{d}, \|\mathbf{k}\|_1 = n} (c_1)^{k_1} \ldots (c_d)^{k_d}.$$